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# A STUDY OF FRACTIONAL q-INTEGRALS

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**Abstract :**
In this paper we have defined two fractional q-operators. Some fundamental properties of these operators are also investigated.

**Key Words :** Fractional q-operators

## 1. INTRODUCTION :

In the recent past a number of authors have studied about the fractional q-integrals. R.P. Agarwal (1) has also studied the fractional q-integral operators. Which was indeed the q-analogue of kober operator. The aim of this paper is to study two new fractional q-integral operators, which are indeed the basic analogues of Saigo's (2) operator and their repeated relations. A theorem is also given in connection with their properties. The results of agarwal are follows one special case.

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**Notations**

For $0 < q < 1$ , let

$(q^{\alpha})_n = (1 - q^{\alpha})\ (1 - q^{\alpha+1})\ \ldots\ (1 - q^{\alpha+n-1}),\ (q^{\alpha})_0 = 1,$

$$\begin{bmatrix} \alpha \\ k \end{bmatrix} = \frac{(1-q^{\alpha})(1-q^{\alpha-1})\ldots(1-q^{\alpha-k+1})}{(q)_k},$$

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = 1$$

We then define Slater (3), the following basic hyper-geometric functions :

$${}_r\phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_r)_n}{(q)_n(b_1)_n\ldots(b_s)_n}.x^n$$

$$e_q(x) = \prod_0^{\infty}[1-xq^n]^{-1} = \sum_0^{\infty}\frac{x^r}{(q)_r},$$

$$(x-y)_v = x^v\prod_0^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{v+n}}\right],$$

$$\Gamma_q(\alpha) = \frac{(1-q)_{\alpha-1}}{(1-q)^{\alpha-1}} \quad (\alpha \neq 0,-1,-2,\ldots)$$

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The basic integrals are defined by

$$\int_x^\infty f(t)d(t:q) = x(1-q)\sum_{k=1}^{\infty} q^{-k} f(q^{-k}x),$$

$$\int_0^x f(t)d(t:q) = x(1-q)\sum_{k=0}^{\infty} q^{k} f(xq^{k})$$

$$\int_0^\infty f(t)d(t:q) = (1-q)\sum_{k=-\infty}^{\infty} q^{k}.f(q^{k})$$

## 2. THE OPERATORS

We now introduced the q-fractional operators

$$I_q^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma_q(\alpha)}.\int_0^x (x-tq)_{\alpha-1}\, {}_2\phi_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{tq}{x}\right) f(t)d(t:q) \qquad ...(2.1)$$

$$= \frac{x^{-\beta}}{\Gamma_q(\alpha)}(1-q)\sum_{k=0}^{\infty} q^{k}(1-q^{k+1})_{\alpha-1}\, {}_2\phi_1(\alpha+\beta,-\eta;\alpha;1-q^{k+1}) f(xq^{k}) \qquad ...(2.2)$$

(2.2) being valid for all α.

This is the q-analogue of Saigo (2) $I_{0x}^{\alpha,\beta,\eta} f$ operator

$$I_{0x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta,-\eta;\alpha,1-\frac{t}{x}\right) f(t)dt$$

The second q-fractional operator is

$$J_q^{\alpha,\beta,\eta} f(x) = \frac{1}{\Gamma_q(\alpha)}.\int_x^\infty (tq-x)_{\alpha-1} t^{-\alpha-\beta}\, {}_2\phi_1\left(\alpha+\beta,-\eta;\alpha,1-\frac{x}{tq}\right) f(t)d(t:q) \qquad ...(2.3)$$

$$= \frac{(1-q)x^{-\beta}}{\Gamma_q(\alpha)}.\sum_{k=1}^{\infty} q^{-k}(q^{-k+1}-1)_{\alpha-1} q^{k(\alpha+\beta)}\, {}_2\phi_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{1}{q^{-k+1}}\right).f(xq^{-k}) \qquad ...(2.4)$$

Which is the q-analogue of Saigo's (2)

$J_{x\infty}^{\alpha,\beta,\eta} f(x)$ operator

$$J_{x\infty}^{\alpha,\beta,\eta} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) M(t)dt \qquad ...(2.5)$$

If β = –α, then (2.1) reduces to the basic analogue of Riemann-Liouville. fractional integral operator

$$R_q^{\alpha} f(x) = \frac{x^{\alpha}}{\Gamma_q(\alpha)}(1-q)\sum_{k=0}^{\infty} q^{k}(1-q^{k+1})_{\alpha-1} f(xq^{k}) \qquad ...(2.6)$$

and if β = 0, then (2.1) reduces to the basic analogue of Erdelyi-Kober fractional integral operator given earlier by R.P. Agarwal. (1)

If β = –α, α > 0 the operator (2.3) reduces to the basic analogue of Weyl fractional integral operator

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$$W_q^{\alpha} f = \frac{x^{\alpha}}{\Gamma_q(\alpha)}(1-q)\sum_{k=1}^{\infty} q^{-k}(q^{-k+1}-1)_{\alpha-1} f(xq^{-k}) \qquad \ldots(2.7)$$

and for $\beta = 0$, (2.3) reduces to the basic analogue of Erdelyi-Kober fractional integral operator.

## 3. REPEATED FRACTIONAL OPERATORS.

(i) $I_q^{\alpha,\beta,\eta}\, I_q^{\gamma,\delta,\alpha+\eta}.f = I_q^{\alpha+\gamma,\beta+\delta,\eta} f$

(ii) $I_q^{\alpha,\beta,\eta}.I_q^{\gamma,\delta,\eta-\beta-\gamma-\delta}.f = I_q^{\alpha+\gamma,\beta+\delta,\eta-\gamma-\delta} f$

(iii) $J_q^{\gamma,\delta,\alpha+\eta}.J_q^{\alpha,\beta,\eta}.f = J_q^{\alpha+\gamma,\beta+\delta,\eta} f$

(iv) $J_q^{\gamma,\delta,\eta-\beta-\gamma-\delta}.J_q^{\alpha,\beta,\eta}.f = J_q^{\alpha+\gamma,\beta+\delta,\eta-\gamma-\delta} f$

To prove 3.(i), we have the L.H.S.

$$= \frac{x^{-\beta-\delta}(1-q)^2}{\Gamma_q(\alpha)\Gamma_q(\beta)}.\sum_{n=0}^{\infty} q^n(1-q^{n+1})_{\alpha-1}\ {}_2\phi_1\left(\alpha+\beta,-\eta;\alpha;1-q^{n+1}\right) f(xq^n)$$

$$.\sum_{k=0}^{\infty} q^{n+k}(q^n-q^{n+k+1})_{\gamma-1}\ {}_2\phi_1\left(\gamma+\delta,-\eta-\alpha;\gamma;1-q^{n+k+1}\right) f(xq^{k+n})$$

$$= \frac{x^{-\beta-\delta}(1-q)^{\alpha+\gamma}}{(1-q)_{\alpha-1}(1-q)_{\gamma-1}}.\sum_{t=0}^{\infty} q^t\ {}_2\phi_1\left(\gamma+\delta,-\eta-\alpha;\gamma;1-q^{t+1}\right)$$

$$.\sum_{n=0}^{\infty} q^{n\gamma}\left(1-q^{n+1}\right)_{\alpha-1}\left(1-q^{t-n+1}\right)_{\gamma-1}\ {}_2\phi_1(\alpha+\beta,-\eta;\alpha;1-q^{n+1}).f(xq^n)$$

$$= \frac{x^{-\beta-\delta}(1-q)}{\Gamma_q(\alpha+\gamma)}.\sum_{t=0}^{\infty} q^t(1-q^{t+1})_{\alpha+\gamma-1}\ {}_2\phi_1\left(\alpha+\gamma+\beta+\delta,-\eta;\alpha+\gamma;1-q^{t+1}\right).f(xq^t)$$

$$= \frac{x^{-\beta-\delta}}{\Gamma_q(\alpha+\gamma)}.\int_0^x (x-tq)_{\alpha+\gamma-1}\ {}_2\phi_1\left(\alpha+\gamma+\beta+\delta,-\eta;\alpha+\gamma;1-\frac{tq}{x}\right).f(t).d(t:q)$$

$$= I_q^{\alpha+\gamma,\beta+\delta,\eta}.f$$

Proceeding on the same line we can prove 3(ii).

For the proof of 3(iii), the L.H.S.

$$= \frac{x^{-\beta-\delta}(1-q)^2}{\Gamma_q(\gamma)\Gamma_q(\alpha)}.\sum_{n=1}^{\infty} q^{-n}(q^{-n+1}-1)_{\gamma-1}.q^{n(\gamma+\delta)}\ {}_2\phi_1\left(\gamma+\delta,-\alpha-\eta;\gamma;1-\frac{1}{q^{-n+1}}\right) f(xq^{-n})$$

$$.\sum_{k=1}^{\infty} q^{-(n+k)}\left(q^{-(n+k)+1}-q^{-n}\right)_{\alpha-1}.q^{(n+k)(\alpha+\beta)}\ {}_2\phi_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{1}{q^{-(k+n)+1}}\right).f(xq^{-(k+n)})$$

$$= \frac{x^{-\beta-\delta}(1-q)^{\alpha+\gamma}}{(1-q)_{\alpha-1}(1-q)_{\gamma-1}}.\sum_{t=1}^{\infty} q^{-t}.q^{t(\alpha+\gamma+\beta+\delta)}\ {}_2\phi_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{1}{q^{-t+1}}\right) f(xq^{-t})$$

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$$.\sum_{n=1}^{\infty}\left(q^{-n+1}-1\right)_{\gamma-1}\left(q^{-(t-n)+1}-1\right)_{\alpha-1}.q^{-(t-n)(\gamma+\delta)-n\alpha}\ {}_2\phi_1\left(\gamma+\delta,-\eta-\alpha;\gamma;1-\frac{1}{q^{-n+1}}\right)f\left(xq^{-n}\right)$$

$$=\frac{x^{-\beta-\delta}(1-q)}{\Gamma_a(\gamma+\alpha)}.\sum_{t=1}^{\infty}q^{-t}\left(q^{-t+1}-1\right)_{\alpha+\gamma-1}.q^{t(\alpha+\gamma+\beta+\delta)}\ {}_2\phi_1\left(\alpha+\gamma+\beta+\delta,-\eta;\alpha+\gamma;1-\frac{1}{q^{-t+1}}\right).f(xq^{-t})$$

$$=\frac{1}{\Gamma_q(\gamma+\alpha)}.\int_x^{\infty}(tq-x)_{\gamma+\alpha-1}t^{-\alpha-\gamma-\beta-\delta}\ {}_2\phi_1\left(\alpha+\gamma+\beta+\delta,-\eta;\alpha+\gamma;1-\frac{x}{tq}\right).f(t)d(t:q)$$

$$=J_q^{\alpha+\gamma,\beta+\delta,\eta}.f$$

Proceeding on the same lines we can prove 3(iv)

On putting $\gamma = -\alpha$, $\delta = -\beta$ in 3(i) ad 3(ii) we get the inverse of $I_q$ ad $J_q$ operators as

$$\left(I_q^{\alpha,\beta,\eta}\right)^{-1}=I_q^{-\alpha,-\beta,\alpha+\eta} \qquad \ldots(3.1)$$

and $$\left(I_q^{\alpha,\beta,\eta}\right)^{-1}=J_q^{-\alpha,-\beta,\alpha+\eta} \qquad \ldots(3.2)$$

**Theorem :**

If $\alpha > 0$, the following results can easily proved.

(i) $I_q^{\alpha,\beta,\eta}f = x^{-\alpha-\beta-\eta}\ I_q^{\alpha,-\alpha-\eta,-\alpha-\beta}f$

(ii) $I_q^{\alpha,\beta,\eta}.x^{\beta-\eta}f = I_q^{\alpha,\eta,\beta}f$

(iii) $J_q^{\alpha,\beta,\eta}\ x^{\alpha+\beta+\eta}f = J_q^{\alpha,-\alpha-\eta,\alpha-\beta}f$

(iv) $J_q^{\alpha,\beta,\eta}.f = x^{\eta-\beta}.J_q^{\alpha,\eta,\beta}f$

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